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EXTERNAL RAYS TO PERIODIC POINTS

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ABSTRACT

We prove that for every polynomial-like holomorphic map P, if $a \in K$ (filled-in Julia set) and the component K_a of K containing a is either a point or a is accessible along a continuous curve from the complement of K and K_a is eventually periodic, then a is accessible along an external ray. If a is a repelling or parabolic periodic point, then the set of arguments of the external rays converging to a is a nonempty closed "rotation set", finite (if K_a is not a one point) or Cantor minimal containing a pair of arguments of external rays of a critical point in $\mathbb{C} \setminus K$. In the Appendix we discuss constructions via cutting and glueing, from P to its external map with a "hedgehog", and backward.

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Introduction

It is known (see [D] and [EL]) that every repelling periodic point of a polynomial P of degree $d \geq 2$ is accessible from its basin of infinity D_P along a curve, and one can choose the curve periodic under P. One of the results of the present paper is that every repelling periodic point of P is accessible along a special curve in D_P: the so-called **external ray** [GM, Appendix A], [LS] (not necessarily periodic under P). We prove also that the set of the arguments of all external rays converging to this point is closed. It is a "rotation" set (i.e. an iterate of $t \to dt (\mathrm{mod}\, 1)$ on it is monotone — but not necessarily strictly monotone) finite or Cantor minimal. In the case $J(P)$, the Julia set of P, is connected, the accessibility along an external ray follows immediately from the accessibility. This is Lindelöf's Theorem, see [CL]. In the case $J(P)$ is not connected this is less simple. We shall also rely on Lindelöf's Theorem but indirectly. (Recently a direct proof of the the accessibility along external rays has appeared [P2].)

We work in fact in a more general setting: of polynomial-like mappings [DH], and lines (leaves) to a Julia set of an f-invariant foliation τ with singularities.

Let us be more precise now. Fix a polynomial P of degree $d \geq 2$. Let D_P denote the basin of infinity

$$D_P = \{z \in \hat{\mathbb{C}}\colon P^n(z) = P \circ \cdots \circ P(z) \to \infty,\ n \to \infty\}.$$

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.

The Julia set $J(P)$ is the closure of the repelling periodic points of P, and for polynomials $J(P)$ coincides with the boundary of D_P. The set $K(P) = \hat{\mathbb{C}} \setminus D_P$ is called the filled-in Julia set. Denote by $u(z)$ Green's function of the domain D_P with the pole at infinity. An **external ray** of P is a gradient line of u, i.e. a trajectory of the vector field grad u which joins infinity with the Julia set or else a limit of such lines. In the former case it is smooth, in the latter it can cross critical points of u. Of course if $J(P)$ is connected, all external rays are smooth because there are no critical points for P in $D_P \setminus \infty$.

The **argument** of an external ray R is the asymptotic normalized angle $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ at which R goes to ∞. The correspondence between external rays and their arguments is one-to-one on the smooth rays and two-to-one on those crossing critical points of u (see Figure 1a).

Given $z \in J(P)$, denote by $\Lambda(z)$ the set of arguments of all rays, which have z as the end point (if such rays exist). So every $t \in \Lambda(z)$ is an **external argument** of z.

It is known [D], [Y], [Mi] that if $J(P)$ is connected, i.e. D_P is simply-connected, then $\Lambda(a)$ *is non-empty and finite* provided a is a repelling or parabolic periodic point of P .

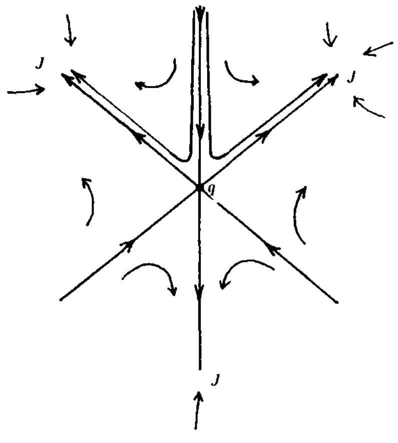

Figure 1a. Two external rays with the same argument.

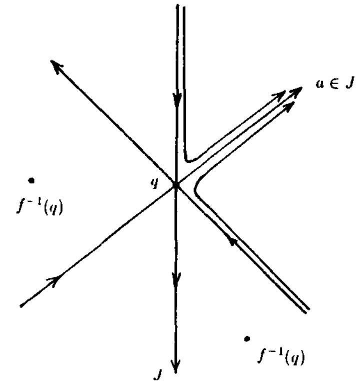

Figure 1b. A pair of critical external rays with arguments in $\Lambda(a)$.

In what follows $J(P)$ is not necessarily connected.

THEOREM 1: *Let a be a repelling or parabolic periodic point for P of period $m(a)$. Then*

1^o *The set $\Lambda(a)$ is a non-empty compact subset of $\mathbb{T}$. It is invariant under the map $\sigma_d^{m(a)}$, where*

$$\sigma_d\colon t \to d \cdot t \ (\mathrm{mod}\, 1),$$

i.e. $\sigma_d^{m(a)}(\Lambda(a)) = \Lambda(a)$.

2^o *If $\Lambda(a)$ is infinite then the ω-limit set $\omega(t_0)$ of the $\sigma_d^{m(a)}$-orbit of every point $t_0 \in \Lambda(a)$ contains a pair of external arguments t, t' of a critical point for $P^{m(a)}$, which belongs to D_P. Moreover, the set $\Lambda(a)$ is a Cantor set and every forward $\sigma_d^{m(a)}$-orbit in $\Lambda(a)$ is dense in $\Lambda(a)$.* See Figure 1b.

3^o *If the component K_a of $K(P)$ containing a is not a one-point set, then $\Lambda(a)$ is finite.*

4^o *If $\Lambda(a)$ contains a periodic point, or is finite, then every $t \in \Lambda(a)$ is periodic under σ_d, of the same period.*

Remark 1: The case when $\Lambda(a)$ is infinite is really possible. For example, it can happen for a fixed point of a quadratic polynomial with disconnected Julia set [GM].

Remark 2: It is not known whether the Cremer fixed point a of a quadratic polynomial is accessible from D_P. But if it is so, then the set $\Lambda(a)$ cannot be closed (Douady, Sullivan), see [Mi]. For recent progress in this direction see [K] and [P-M]. As is shown there, the critical point of a quadratic polynomial with the Cremer fixed point is *not accessible* from the basin of infinity D_P.

From Theorem 1 we obtain immediately

COROLLARY 1: *The total number of the repelling and parabolic periodic orbits* $\mathrm{Orb}(z)$ *of* P *for which* $\Lambda(z)$ *is infinite is bounded by the number of critical values of* P *in* D_P.

The assertion of Theorem 1, that $\Lambda(a) \neq \emptyset$, will be concluded from the following

THEOREM EL: *Every repelling or parabolic periodic point* $a \in J(P)$ *is accessible from* D_P *along a curve* l *(this means a continuous map* $l : (0,1] \to \mathrm{cl}(D_P)$ *such that* $l((0,1)) \subset D_P$ *and* $l(1) = a$*), such that* $P^n(l) = l$ *for some* $n > 0$.

This was proved in the case of every non-connected $J(P)$ for every repelling periodic point a in [EL] and for parabolic points in [E] and in [P2].

To deduce Theorem 1 from Theorem EL we shall prove the following

THEOREM 2: *Let* $a \in J(P)$. *Assume that either the component* K_a *of* $K(P)$ *containing* a *is a point or* a *is accessible along a curve* l *and* K_a *is eventually periodic (i.e. there exist integers* $n \geq 0$, $k > 0$ *such that* $P^{n+k}(K_a) = P^n(K_a)$*). Then the point* a *is accessible along an external ray, which is equivalent in* $\mathbb{C} \setminus K_a$ *to* l. *In the case* $K_a = \{a\}$, *the set of the arguments of all such rays is closed.*

We mean here that two curves in a domain U which converge to the same $z \in \partial U$ are **equivalent** in U if they are homotopic in U in such a way that all the curves along the homotopy converge to z.

In [P2] the accessibility of *good* points is proved. The assumption of being *good* is quite weak, all periodic repelling and parabolic points are *good*, moreover almost every point for an arbitrary P-invariant measure of positive Lyapunov exponents is *good*. So the assertion $\Lambda(a) \neq \emptyset$ holds by Theorem 2 for every *good* a (such that K_a is eventually periodic).

Let us mention that in [P2] we prove directly that $\Lambda(a) \neq \emptyset$ for every *good* a.

If $J(P)$ is connected then D_P is conformally equivalent to the disc $D_* = \{|z| > 1\}$ in the Riemann sphere, and one can choose a conformal isomorphism $B\colon D_P \to D_*$ in such a way that $B \circ P = P_0 \circ B$ and $P_0(z) = z^d$. The curve $B(l)$ in D_* ends at a point $z_0 \in S^1$. By Lindelöf's Theorem the external ray $B^{-1}(\{\rho z_0\colon \rho > 1\})$ also converges to the point a. If a is periodic then, as mentioned above, we can find l such that $P^n(l) = l$ for some $n > 0$. Hence z_0 is periodic for P_0. The rest of the assertion of Theorem 1, that there exist only finitely many external rays converging to a, now readily follows from the fact that P preserves the order of the external rays converging to the point a [Mi, Lemma 18.3].

We are going to use the same ideas in the general case of the non-connected Julia set. Theorem 1 will be proved in a more general setting, namely, for polynomial-like mappings and τ-external rays as mentioned at the beginning of the paper. We call then Theorem 1: **Theorem 1^τ** and Theorem 2: **Theorem 2^τ**.

The exact definitions will be provided in Section 1. Riemann mapping will be replaced by a map from the exterior of a *hedgehog* as in [LS], in the annulus of the external map for our polynomial-like map [DH]. An alternative way: constructing the external map annulus by some surgery on the original Riemann surface, a neighbourhood of the filled-in Julia set, will be discussed in the Appendix.

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1. External rays and angles

According to Douady and Hubbard [DH], a polynomial-like mapping of degree $d \geq 2$ is a triple (U, U_1, f), where U and U_1 are open subsets of $\mathbb{C}$ isomorphic to discs, with U_1 relatively compact in U, and $f\colon U_1 \to U$ is a complex analytic and a branched covering mapping of degree d.

A compact

$$K(f) = \bigcap_{n=1}^{\infty} f^{-n}(U)$$

is called a **filled-in Julia set**. The **Julia set** of f is $J(f) = \partial K(f)$. Denote by C_f the set of all critical points of f in $U_1 \setminus K(f)$ and $C_f(\infty) = \bigcup_{n=0}^{\infty} f^{-n}(C_f)$. The filled-in Julia set is connected if and only if C_f is empty.

In the theory of polynomial-like mappings the **external map of** f plays an important role. It is a real analytic expanding map $h_f\colon S^1 \to S^1$ of degree d, which is constructed as follows [DH]:

Let W be an open topological disc with closure in U with smooth boundary, such that $W_1 = f^{-1}(W)$ is also a topological disc (and hence $C_f \subset W_1$), with closure in W. Let $L \subset W_1$ be a compact topological disc in W_1 containing $f^{-1}(\overline{W_1})$ and all the critical points of f. Let an annulus X_n be a covering space of the annulus $X_0 := W \setminus L$ of degree d^n, $\rho_n\colon X_{n+1} \to X_n$ and $\pi_n\colon X_n \to X_0$ be the projections and let X be the disjoint union of the X_n. For each n choose a lifting

$$\tilde{f}_n\colon T_n = \pi_n^{-1}(W_1 \setminus L) \to X_{n+1},$$

of f so that $\rho_n \circ \tilde{f}_n = \tilde{f}_{n-1} \circ \rho_{n-1}$, see Figure 2. Let now A^* be the quotient of X by the equivalence relation identifying x to $\tilde{f}_n(x)$ for all $x \in T_n$ and all $n = 0, 1, \ldots$. The open set A_1^* is the union of the images of the X_n, $n = 1, 2, \ldots$ under this identification, and $h_f\colon A_1^* \to A^*$ is the holomorphic map induced by ρ_n's.

The Riemann surface A^* is isomorphic to an annulus of finite modulus, say $\log w$. So we can identify A^* with the annulus $\{z\colon 1 < |z| < w\}$ and use the Schwarz reflection principle to extend h_f to an expanding map h_f of S^1 (cf. e.g. [P1, Sect.7], we shall use this observation again in Proof of Lemma 2.1). By **expanding** we mean here $|(h_f^m)'| > 1$ for an integer $m > 0$. This is the external map of f. It is defined up to a real analytic homeomorphism (change of coordinates) on the unit circle.

We use this construction to define τ-rays and their arguments as follows: Denote by B_f a conformal isomorphism between neighbourhoods of the closures of

$$A_0 = W \setminus W_1$$

and

$$A_0^* = A^* \setminus A_1^*$$

in W and A^* respectively, such that B_f conjugates f and h_f in W_1 near the boundary of W_1. This is just the identity if we remember that $W \setminus L = X_0$ is a part of A^*. Let us fix a smooth foliation τ in a neighbourhood of the closure of A_0 invariant under f, and consider the foliation $\tau^* = B_f(\tau)$ in a neighbourhood of the closure of A^*. Assume that each leaf of τ joins the outer boundary of A_0 (i.e. ∂W) to the inner one and intersects the boundaries transversally. Extend τ and τ^* to new τ and τ^* on $W \setminus K(f)$ and A^* respectively by taking $f^{-n}(\tau), h_f^{-1}(\tau^*), n = 1, 2, \ldots$. Then τ^* is a smooth foliation, while τ is a smooth foliation with singularities at $C_f(\infty)$.

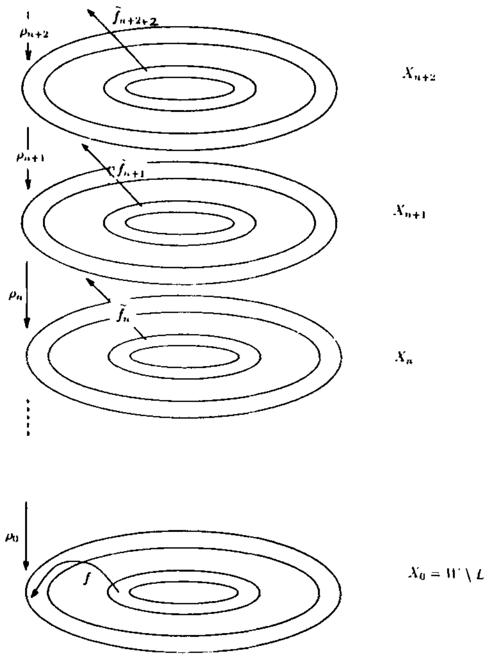

Figure 2. Douady–Hubbard's construction of the external map.

Fix the orientations of τ, τ^* positive towards $K(f)$, respectively S^1. The oriented leaves of these foliations are called τ-lines, or τ^*-lines resp. The initial point of every τ-line is either a point of ∂W or some point from $C_f(\infty)$. In the former case the maximal τ-line is the τ-**radius**, in the latter case it is the τ-**cut**. We assume that the initial point of the τ-cut belongs to it.

We define a **smooth** τ-**ray** as a τ-radius which converges to $J(f)$. If a τ-radius does not converge to $J(f)$ then it ends at a point of $C_f(\infty)$. In this case a τ-**ray** (not smooth) containing the τ-radius is defined as a limit of smooth τ-rays, see [GM, Appendix A]. Of course there are two τ-rays containing the τ-radius, the limits from both sides of it. In the sequel we shall give another equivalent definition of τ-rays and define also their arguments with the help of the annulus A^*.

We call each leaf of τ^* in the annulus A^* a τ^***-ray**. Every point x in A^* belongs to one and only one τ^*-ray, which we will denote by $\tau^*(x)$.

We extend the map B_f to map B_f^τ as follows (cf. [LS], [SN]). The domain A_τ of B_f^τ is by definition W without $K(f)$ and the τ-cuts. This domain is a topological annulus because it is the union of the τ-radii. Indeed one takes a smooth vector field $V(\tau)$ on $W \setminus K(f)$ tangent to τ, with zeros precisely at $C_f(\infty)$, directed towards $K_f(\infty)$. Then one parametrizes A_τ by the initial points of the τ-radii and the time along them given by the flow generated by $V(\tau)$.

Note that $W \setminus W_1$ is in A_τ. Now let $z \in A_\tau$ and $\tau(z)$ be the τ-radius passing through z. Then $B_f^\tau(z)$ is defined as the only point $x \in A^*$ such that the part of $\tau(z)$ in $W \setminus W_1$ is mapped by B_f to the part of $\tau^*(x)$ in $A^* \setminus A_1^*$, and

$$B_f(f^{n(z)}(z)) = h_f^{n(z)}(x), \tag{1.1}$$

where $n(z)$ is the integer such that $f^{n(z)}(z) \in W \setminus W_1$

The map B_f^τ is injective and it is a holomorphic continuation of B_f along the τ-radii, hence, it is a conformal isomorphism of A_τ onto $U_\tau \subset A^*$. It conjugates $f|_{A_{1,\tau}}$ and $h_f|_{U_{1,\tau}}$, where $A_{1,\tau} = A_\tau \bigcap W_1$ and $U_{1,\tau} = U_\tau \bigcap A_1^*$.

The domain U_τ is called a **hedgehog-like annulus.** The inner boundary S_τ of U_τ is called a **hedgehog**. It consists of the unit circle and the set of **needles**. We shall explain it below.

Let us begin with defining the τ**-external angle (argument)** of a τ-radius R of f. The image $B_f^\tau(R)$ is contained in a τ^*-ray $R^* \subset A^*$. This R^* ends at a unique point $e^{2\pi i t}$, $t \in \mathbb{T}$, of the unit circle S^1. The limit set of R^* is a point, because h_f is an expanding map of the subannulus A_1^* and, hence, the Euclidean length of R^* is finite (a similar assertion is proved in detail in Proof of Lemma 2.1.1^o). We will call t the τ-external argument $\arg_{B_f^\tau}$ of the τ-radius R (and its points) in the dynamical plane $z \mapsto f(z)$, and the τ-argument $\arg_\tau$ of the curve R^* (and its points) in A^* (i.e. in the uniformization plane $z \mapsto h_f(z)$):

$$t = \arg_{B_f^\tau}(R) = \arg_\tau(R^*).$$

In particular, we have defined the argument $\arg_\tau(x)$ of every point $x \in A^*$.

Define N_x as the part of the leaf $\tau^*(x)$ starting from x to S^1.

Every point $q \in C_f$ is a common terminal point of a finite set of τ-radii of f. Denote by $\Lambda_\tau(q)$ the set of the τ-external arguments t of the corresponding radii R_t with the end at $q \in C_f$. For $t \in \Lambda_\tau(q)$, there exists

$$\lim_{z \to q z \in R_t} B_f^\tau(z) = \tilde{B}^\tau(q,t)$$

(this equality is also a notation).

The hedgehog S_τ consists of the unit circle S^1 and the set of all needles N_x, see Figure 3, [LS]:

$$S_\tau = S^1 \cup \bigcup_{q \in C_f} \bigcup_{t \in \Lambda_\tau(q)} \bigcup_{n=0}^{\infty} \bigcup_{h_f^n(x) = \tilde{B}^\tau(q,t)} N_x.$$

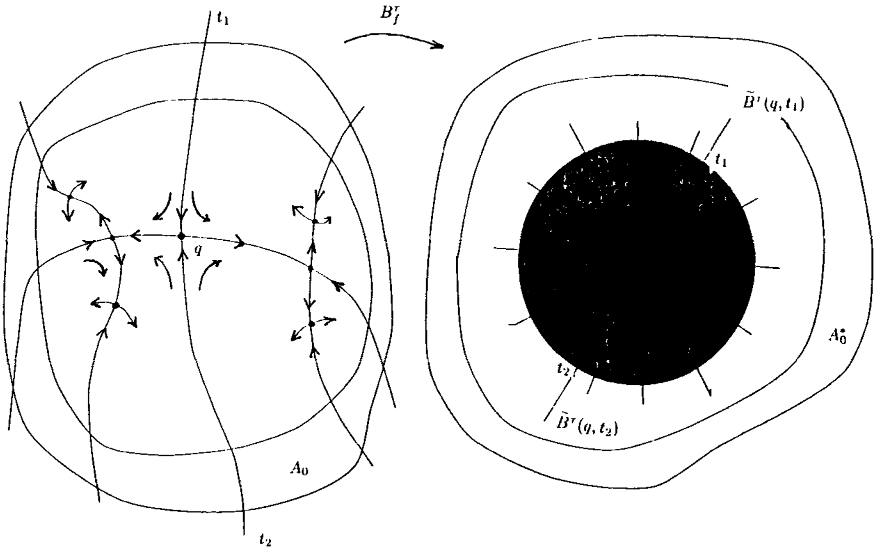

Figure 3. The hedgehog.

Let us recall that we defined each smooth τ-ray in the dynamical plane as a τ-radius R which extends up to the Julia set $J(f)$ (i.e. R does not end at a point of $C_f(\infty)$). Let the end point of R be now a point of $C_f(\infty)$. Then $B_f^\tau(R)$ lands at the top x of a needle N_x. The function $(B_f^\tau)^{-1}$ extends to two continuous functions B_+ and B_- on two banks of N_x. (In other words, we cut along the needle and extend the map to the sides of the cut.) To see it formally use the formula (1.1) and the fact that no point of a needle except points in S^1 is a limit of other needles, as their lengths shrink to 0.

This allows us to define the right and the left τ-rays (non-smooth) containing the τ-radius R:

$$R_t^{\pm} = B_{\pm}^{-1}(\{\xi: 1 < |\xi| < \infty,\ \arg_\tau(\xi) = t\}),$$

where $t = \arg_\tau(x)$ (this is discussed in more detail in the Appendix).

Denote this extension of $(B_f^\tau)^{-1}$ by $\hat{B}$.

The **argument** of a τ-ray is the argument of a τ-radius which is contained in the τ-ray.

It is comfortable to assume about τ the following:

Fidelity assumption. Different τ^*-rays end at different points in S^1 (i.e. the correspondence between τ^*-rays and the external angles is 1-to-1).

Remark 1.1: In Theorem 1 we have a special case, where τ is the foliation into trajectories of the gradient flow for Green's function u on D_P for a polynomial P. Indeed then $f(\tau) = \tau$ (in the sense the image of each leaf is contained in a leaf to take care of singularities) because $u \circ f = d \cdot u$ and f is holomorphic. Observe that the fidelity assumption holds. This is so because τ^* is a foliation into gradient lines for a harmonic function being the extension of $u \circ B_f^{-1}$ on $A^* \setminus A_1^*$ to A^*. This function converges to 0 on points converging to S^1 so it extends harmonically beyond S^1. Hence τ^* extends smoothly beyond S^1.

Having given a polynomial-like f and $W \supset W_1$, $A^* \supset A_1$ as above one can always find τ satisfying the fidelity. For example, one can conjugate h_f to $z \mapsto z^d$ starting with an arbitrary smooth conjugacy Φ on a neighbourhood of $W \setminus W_1$ to a neighbourhood of $\{\theta \leq |z| < \theta^d\}$, $\theta > 1$ and extending it by $(\Phi \circ h_f^n)^{1/d^n}$. Then one defines $\tau^* = \Phi^{-1}$(the foliation into radii). So τ^* extends through S^1, though it is usually not smooth there.

Observe finally that the fidelity is not always true. For example, join a point $z \in \partial W$ to $z_1 \in \partial W_1, f(z_1) = z$ and $w_1 \in \partial W_1$, $w_1 \neq z_1$ close to z_1 to $f(w_1)$. Then τ^*-leaves through $B_f(z)$ and $B_f(w)$ end at the same point in S^1.

Definition 1.1: The set of external arguments is the unit circle, but to every base of a needle there correspond two τ-rays (assuming the fidelity). To have a one-to-one correspondence between the external rays and their arguments one should cut the unit circle at all the bases of the needles. Then each external argument, base of a needle, in the above sense gives rise to external arguments of the rays R^+ and R^-: t^+ and t^-. This is Definition A.1 from [GM].

We denote the set of all the external arguments in the above sense, i.e. the result of cuttings of $\mathbb{T}$, by $\hat{\mathbb{T}}$. We denote the projection of $\hat{\mathbb{T}}$ to $\mathbb{T}$ by $\hat{P}$. If we consider an external argument in $\hat{\mathbb{T}}$ rather than in $\mathbb{T}$ we call it the external **ˆargument**.

Definition 1.2: We define a **level function** M as an arbitrary smooth function $W \setminus K(f)$ which is 0 on ∂W, 1 on ∂W_1 is strictly increasing on the leaves of τ and $M(f(z)) = M(z) - 1$ for $z \in W_1 \setminus K(f)$. (The letter M is from the analogy with the Morse function.)

Let Γ denote the foliation (with singularities at $C_f(\infty)$) of $W \setminus K(f)$ into the components of the constant M.

From our definitions we deduce the following

LEMMA 1.1: *The mapping from $\hat{\mathbb{T}} \times \mathbb{R}^+$ to $W \setminus K(f)$, mapping each (t, r) to the point on the τ-ray with the ˆargument t and the value of the function M equal to r, is continuous.*

Remark 1.2: We remind the reader again that the external arguments of τ-rays (as well as the external map h_f itself) are defined only up to a real analytic homeomorphism of the unit circle. In particular there is no reason that external arguments of preperiodic points are rational. If one wants a canonical description of external arguments one should either conjugate the external map to $z \to z^d$ but the change of coordinates would be usually not smooth, or change coordinates on S^1 to make the length measure invariant and 1 a fixed point. This defines external arguments mod $\frac{1}{d-1}$.

2. External rays to a component: Proof of Theorem 2

Let $z \in J(f)$. Denote by $\Lambda_\tau(z)$ the set of the τ-external arguments $t \in \mathbb{T}$ of the τ-rays R_t, such that the z is the unique limit point of R_t on $J(f)$ (landing point of R_t). Similarly define the set $\hat{\Lambda}_\tau(z) \subset \hat{\mathbb{T}}$ (see Remark 1.2).

Remark 2.1: Though two external rays can have the same argument $t \in \mathbb{T}$, this happens only if they both contain a point $c \in C_f(\infty)$. Then they converge to $J(f)$ in different components of $\mathbb{C} \setminus \Gamma(c)$ ($\Gamma(c)$ is the union of the leaves of Γ containing c in the closures). So they cannot both land at the same z, hence the notation R_t makes sense. In other words the correspondence $\hat{P}$ between $\hat{\Lambda}_\tau(z)$ and $\Lambda_\tau(z)$ is one-to-one.

As mentioned in Remark 2 of the Introduction, the set $\Lambda_\tau(z)$ can be empty. In any case, it is of zero Lebesgue measure (otherwise take a point of density and use the fact that the external map is expanding). Suppose the compact set $J_\tau(f)$ (the Julia set of f plus the τ-cuts) is locally connected (it is the case, if $f\colon J(f) \to J(f)$ is expanding). Then $\Lambda_\tau(z)$ is a non-empty compact set for every $z \in J(f)$ (to prove this, use the map $\hat{B}$ its definition precedes the definition of the fidelity in Section 1). In the general case we do not know whether the compactness is always true (see Remark 2 in Introduction), but it is true if $\{z\}$ is a component of $J(f)$.

PROPOSITION 2.1: *Let x be a point, such that the single-point set $\{x\}$ is a component of the Julia set $J(f)$. Then the sets $\Lambda_\tau(x)$ and $\hat{\Lambda}_\tau(x)$ are non-empty and compact.*

Proof: There exists such a sequence K_i of open sets bounded by leaves of Γ that $\overline{K_{i+1}} \subset K_i$ and

$$\bigcap_{i=1}^{\infty} K_i = \{x\}.$$

Let $y_i = \{R_t\}$ be the set of τ-rays, such that each of them crosses the boundary ∂K_i. Every such a ray has exactly one intersection with ∂K_i. Let $T_i = \{t\colon R_t \in y_i\}$. Evidently, $T_{i+1} \subset T_i$ and T_i is a non-empty compact subset of $\hat{\mathbb{T}}$ (by Lemma 1.1) and $\bigcap_{i=1}^{\infty} T_i = \hat{\Lambda}_\tau(x)$ so it is also non-empty and compact. Hence the same holds for $\Lambda_\tau(x) = \hat{P}(\hat{\Lambda}_\tau(x))$. This ends the proof.

Remark 2.2: Observe that it does not matter whether we prove the compactness of $\hat{\Lambda}_\tau(x)$ or of $\Lambda_\tau(x)$, because the compactness of the latter implies the compactness of the former. Indeed if $\hat{P}(t_n) \to \hat{P}(t_0)$ for $t_n \in \hat{\Lambda}_\tau(x)$, then t_0 cannot be on the other side of its adjacent gap with respect to t_n's because in such a case its ray would converge to a different component of $J(f)$ than rays of t_n's. They could not converge to the same x.

Let K be a component of $K(f)$. Choose a conformal isomorphism Φ from the double-connected domain $W_1 \setminus K$ onto an annulus

$$A_K = \{z\colon 1 < |z| < w(K)\}.$$

The non-singular leaves of the foliation Γ and τ-rays are mapped under Φ to two families of curves. We will call those of them which either surround the unit circle

S^1 or have the limit set in S^1, K-related leafs and K-related rays, respectively, see Figure 4. Every K-related ray has the argument, which is just the argument of the corresponding τ-ray in W. The same concerns ˆarguments.

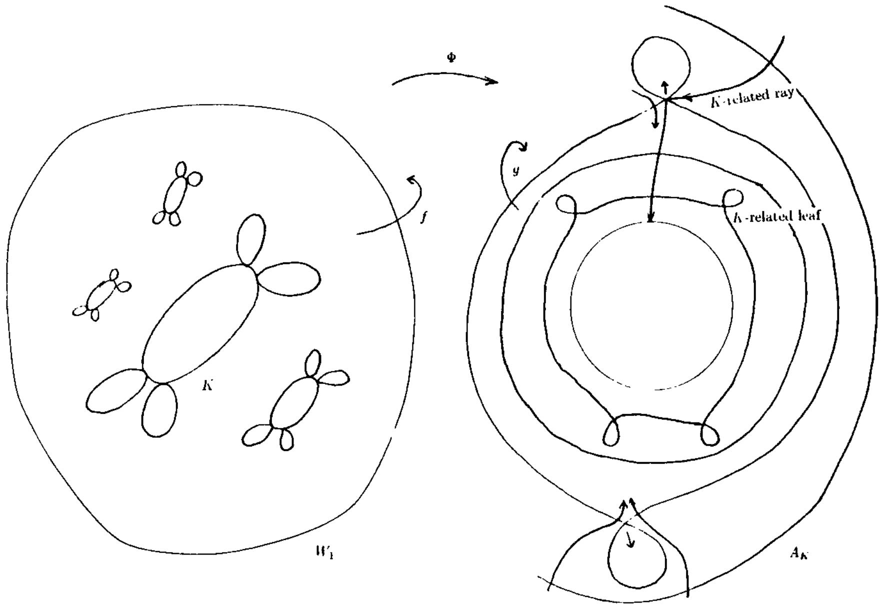

Figure 4. Uniformization coordinates outside K.

Remark 2.3: Let R be a τ-ray (in $W \setminus K(f)$). If it has a limit point in the component K, then all its limit points belong to K since they form a continuum. Therefore, if a limit point of the curve $\tilde{R} = \Phi(R)$ belongs to S^1 then all limit points of this curve belong to S^1, that is, $\tilde{R}$ is a K-related ray.

Remark 2.4: Observe that the set of K related rays is non-empty and the sets of the corresponding external arguments or ˆarguments are compact. The proof is the same as Proof of Proposition 2.1.

LEMMA 2.1: *Suppose that K is not one-point and that it is eventually periodic under f. Then*

1^o *Every K-related ray has finite length and, hence, converges to a unique point of S^1. The lengths of the parts of the rays where the function $M \circ \Phi^{-1} \geq t$ converge to 0 uniformly (exponentially fast) as $t \to \infty$.*

2^o *The sets $\lambda(z_0)$ and $\hat{\lambda}(z_0)$ of arguments and ˆarguments of all K-related rays converging to a given point z_0 of S^1 are non-empty compact in $\mathbb{T}$*

or $\hat{\mathbb{T}}$.

3° *The set of all* K*-related rays in* $\{z\colon 1 < |z| < 1+\epsilon\}$ *converging to a point* z_0 *lies in an angle*

$$\{z\colon |\arg(z-z_0) - \arg z_0| \le \alpha\},$$

where $\alpha \in (0, \pi/2)$ *and* ϵ *do not depend on* $z_0 \in S^1$.

Proof: 1° Passing to an iteration and to an f^n-image we can assume that $f(K) = K$. Let, as above, $A_K = \Phi(W \setminus K)$, $A_{1,K} = \Phi(W_1 \setminus K)$ and

$$g = \Phi \circ f \circ \Phi^{-1} : A_{1,K} \to A_K$$

be a conjugated map. For a leaf $\gamma_0 \in \Gamma$ that surrounds K, there is a component γ_1 of its f-preimage that also surrounds K and lies in the component $U(K)$ of $\mathbb{C} \setminus \gamma_0$ containing K. If we denote the component of $\mathbb{C} \setminus \gamma_1$ containing K by $U(K)_1$ and γ_0 is chosen sufficiently close to K that there are no critical points of f in $U(K) \setminus K$, we have

$$K = \bigcap_{n \ge 0} (f|_{U(K)_1})^{-n}(U(K)).$$

The set K is not a point by our assumption. Hence by [P1, Section 7] the map g extends to an expanding holomorphic map in an annulus

$$U_0 = \{z\colon 1 - \rho_0 < |z| < 1 + \rho_0\},$$

for some $\rho_0 > 0$.

That means that after passing, if necessary, to an iterate of g (which we also denote g) we have

$$|(g^{-1})'(z)| < a < 1 \tag{2.1}$$

for every $z \in U_0$ and for every branch g^{-1} such that $g^{-1}(z) \in U_0$.

Fix a K-related leaf $\tilde{\gamma}_0 \subset U = A_K \cap U_0$. Then, for each $n = 1, 2, \ldots$, $\tilde{\gamma}_n = \{z \in U\colon g^n(z) \in \tilde{\gamma}_0\}$ is also a K-related leaf. Denote by l_n the supremum of lengths over all the arcs of the K-related rays joining $\tilde{\gamma}_n$ and $\tilde{\gamma}_{n+1}$, $n = 0, 1, \ldots$.

Then (2.1) gives us: $l_n < a^n \cdot l_0$. Given a K-related ray, its length in the component of $\mathbb{C} \setminus \tilde{\gamma}_0$ containing S^1 is bounded from above by

$$\sum_{n=0}^{\infty} a^n l_0 < \infty,$$

and its length in the component bounded by $\tilde{\gamma}_t$ is bounded by $\sum_{n=t}^{\infty} a^n l_0$ which converges to 0 as $t \to \infty$.

2^o Fix a closed arc $I \subset S^1$. First, there exists a K-related ray converging to a point of I. Otherwise no K-related ray ends in the interval $g^n(I)$, for every n. This is impossible because $g^n(I) = S^1$ for big n and the set of K-related rays is non-empty (see Remark 2.4). Second, we want to show that the set $\hat{\lambda}(I)$ of the ˆarguments of all K-related rays ending in I is closed in $\hat{\mathbb{T}}$. It will imply the statement since $\hat{\lambda}(z_0) = \bigcap \hat{\lambda}(I_1)$ over all closed intervals I_1 covering z_0. It also implies the assertion on $\lambda(z_0)$ by the projection $\hat{P} : \hat{\lambda}(z_0) \to \lambda(z_0)$.

The compactness of $\hat{\lambda}(I)$ follows from Lemma 1.1 (continuity), the continuity of Φ and from the uniform convergence in the assertion 1 of our Lemma 2.1. $\hat{\lambda}(I)$ is closed as the preimage, for a continuous function, of the compact set I.

3^o By the Koebe distortion theorem one can choose $0 < \rho < \rho_0$ such that for every

$$z \in U_1 = \{z \colon 1 - \rho < |z| < 1 + \rho\}$$

every $n = 1, 2, \ldots$ and every holomorphic branch g^{-n}

$$\left| \frac{(g^{-n})'(x)}{(g^{-n})'(y)} \right| < 2 \tag{2.2}$$

whenever $|z - x| < \rho$ and $|z - y| < \rho$.

Introduce the following notations:

Given $x \in U_1$, denote by l_x the part of the K-related ray passing through x between x and S^1 (if such a ray exists). This notation is correct: if another K-related ray passes through x and next ramifies from l_x, it goes to a component of $\Phi(J(f))$, not to S^1. So it is not K-related. (The same argument was used already at the beginning of this Section.)

Denote by h_x the interval which joins x and S^1, orthogonal to S^1. Denote by $l(x)$ and $h(x)$ the corresponding Euclidean lengths. We can assume about the K-related leaf $\tilde{\gamma}_0$ in U_1 that

$$l(x) < \rho \text{ for all } x \text{ between } \tilde{\gamma}_0 \text{ and } S^1. \tag{2.3}$$

Set $\tilde{\gamma}_1 = g^{-1}(\tilde{\gamma}_0)$. There exists a positive β_0 less than 1 such that

$$\frac{h(x)}{l(x)} > \beta_0 \tag{2.4}$$

for all points x in the annulus V between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$.

Fix the maximal $\epsilon_0 > 0$ such that

$$U_2 = \{z \colon 1 - \epsilon_0 < |z| < 1 + \epsilon_0\}$$

does not intersect $\tilde{\gamma}_1$. We intend to prove the assertion 3^o of our Lemma with

$$\alpha = \arccos\left(\frac{\beta_0}{8L}\right),$$

where $L = \sup|g'|$ is a Lipschitz constant for g, and with ϵ between 0 and ϵ_0 so small that $1 < |z| < 1+\epsilon$ and $h(z)/|z-z_0| \geq 2\cos\alpha$ implies $|\arg(z-z_0) - \arg z_0| \leq \alpha$.

It is enough to prove that

$$\frac{h(x)}{l(x)} > \beta = \frac{\beta_0}{4L}$$

for all $x \in U$. Assume the contrary: there exists $x_* \in U$,which belongs to some K-related leaf $\tilde{\gamma}_*$ with

$$h(x_*)/l(x_*) \leq \beta. \tag{2.5}$$

Choose the first n such that $g^n(x_*) \in V$ (that is, the K-related leaf $g^{kn}(\tilde{\gamma}_*)$ lies between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$).

The lengths $h^{(i)}$ and $l^{(i)}$ of the curves $g^i(h_{x_*})$ and $g^i(l_{x_*})$ cannot exceed ρ for all $i = 0, 1, \ldots, n$. This holds for $l^{(i)}$ by (2.3), because $g^i(x_*)$ is between $\tilde{\gamma}_0$ and S^1. We cope with $h^{(i)}$'s by induction: $(h^{(0)}) < \rho$ by the definition of U_1. If it holds for all $i \leq j-1$, then by (2.2)

$$\frac{h^{(j-1)}}{l^{(j-1)}} \leq 4 \cdot \beta = \beta_0/L.$$

Then

$$h^{(j)} \leq Lh^{(j-1)} \leq \beta_0 \cdot l^{(j-1)} < l^{(j-1)} < \rho.$$

Now we use the assumption (2.5) and again apply (2.2), and we obtain for $z_* = g^n(x_*) \in V$,

$$\frac{h(z_*)}{l(z_*)} \leq \frac{h^{(n)}}{l^{(n)}} \leq 4 \cdot \beta = \beta_0/L < \beta_0.$$

This contradicts (2.4). Lemma 2.1 is proved.

Proof of Theorem 2 (and 2^τ): The case $\{z\}$ is a component of $K(f)$ was considered in Proposition 2.1. It remains to consider the case of an eventually periodic component K containing z. We have assumed that a curve $l \subset W \setminus K(f)$ converges to the point $z \in J(f)$. Hence, the curve $\tilde{l} = \Phi(l)$ converges to a point $\tilde{z} \in S^1$ and the limit of the function Φ^{-1} along the curve $\tilde{l}$ exists and equals z. By Lemma 2.1.3° we can apply Lindelöf's Theorem (on the existence of the nontangential limit) to every K-related ray ending at the point $\tilde{z}$. By Lemma 2.1.2° at least one such ray exists, which completes the proof.

3. Proof of Theorem 1 (and 1^τ)

LEMMA 3.1: *Suppose that $h\colon S^1 \to S^1$ is a continuous expanding map, i.e. there exist $\eta > 0$, $\xi > 1$ and $k > 0$ such that if $x, y \in S^1$, $x \neq y$, $\operatorname{dist}(x,y) < \eta$, then $\operatorname{dist}(h^k(x), h^k(y)) > \xi \cdot \operatorname{dist}(x,y)$. Let A be an h-invariant subset of S^1, namely $h(A) = A$, on which h is monotone. Monotone means that for every $x, y, z \in S^1$ such that y is strictly between x and z, in the standard orientation on S^1, $h(y)$ is between $h(x)$ and $h(z)$. (We allow $h(y) = h(z)$ or $h(x)$. If $h(x) = h(z)$ then the condition is empty; we allow $h(y)$ to be anywhere.)*

(Later, for y between or strictly between x, z, the notation $x < y < z$ or $x \leq y \leq z$ will be used. Similarly we shall consider $x < y \leq z$ and $x \leq y < z$.)

Suppose also that

(i) *there exists an integer $m > 0$ and a point $x_0 \in S^1$, such that for every $x, y \in A$ if $x_0 < x < y < x_0$ then $x_0 < h^m(x) \leq h^m(y) < x_0$;*

(ii) *if $x, y \in A$, $x \neq y$ and $h(x) = h(y)$, then neither x nor y is periodic for h.*

Then every point in A is periodic of the same period m. A contains at most $d^m - 1$ points, where d is the degree of h.

Remark 3.1: This is slightly stronger than [Mi, Lemma 18.3] where strict monotonicity is assumed. Here we obtain the strict monotonicity only *a posteriori* in the assertion of Lemma. Remark that we do not assume the compactness of A, unlike in Lemma 3.2 below. Instead we have the point x_0 playing a *separating* role. This implies *rotation number* on A being rational, cf. Remark 3.3.

Proof of Lemma 3.1: The proof is the same as in [Mi]. Let us recall it: We pass to an iterate so that we can assume that $m = 1$. Let $x, h(x) \in A \setminus \{x_0\}$. If $h(x) \neq x$, say $x_0 < x < h(x) < x_0$, then by induction:

$$x_0 < x < h(x) < h^2(x) < \cdots < h^n(x) < \cdots < x_0. \tag{3.1}$$

Otherwise $h^{n-1}(x) < h^n(x) = h^{n+1}(x)$, which contradicts (ii).

Observe finally that (3.1) contradicts the expanding property of h.

We can cope now with $x = x_0$, setting as a new x_0 the fixed point we have just found. The situation A consists only of x_0 and its h-preimages different from x_0 is excluded by the assumption h maps A onto A.

Recall also Douady's Lemma (see for example [Mi, Lemma 18.8]):

LEMMA 3.2: *If h is a continuous expanding map on a compact metric space and A is a compact h-invariant set on which h is a homeomorphism, then A is finite.*

Proof of Theorem 1 (and 1^τ):

PROOF OF 1^o: The $h_f^{m(a)}$-invariance of $\Lambda_\tau(a)$ follows from $f^{m(a)}(a) = a$ and the facts that:

(1) the h_f-image of a τ-ray (restricted to W_1) is a τ-ray and

(2) the $f^{m(a)}$-preimage chosen close to a according to the branch of $f^{-m(a)}$ fixing a extends to a τ-ray (not always uniquely), so h_f maps $\Lambda_\tau(a)$ "onto" $\Lambda_\tau(a)$.

The non-emptyness and compactness of $\Lambda_\tau(a)$ in the case $K_a = \{a\}$ has been already discussed in Proposition 2.1 and Theorem 2. Here assume that K_a is strictly larger than $\{a\}$. Of course K_a is periodic as containing the periodic point a. By the accessibility of a: Theorem EL, and by Theorem 2, we know that the set $\Lambda_\tau(a)$ is non-empty. Let us prove that this set is closed (in $\mathbb{T}$). Introduce a subset X of S^1 as follows. A point $x \in X$ iff there exists a τ-ray R converging to the point a such that the K_a-related ray $\Phi(R)$ converges to x. Taking into account Lemma 2.1.2^o, it is enough to show that the set X is finite. Note that X contains a periodic point of the map g. This follows from the fact that the point a admits a periodic (under f) access l, see again Theorem EL. Hence the curve $\Phi(l)$ ends at a periodic for g point $b \in S^1$. Now the finitness of

X follows from Lemma 3.1 for $h := g, A := X, x_0 := b$. In fact we can refer to [Mi, Lemma 18.3] because we know *a priori* that g is stricly monotone on X, as $f^{m(a)}$ is a local homeomorphism around a.

Let us prove this strict monotonicity: For each $x, y \in X$ observe that $x \neq y$ is equivalent to the property that for each K-related ray $R(x)$ and $R(y)$ ending at x, y respectively, $\Phi^{-1}(R(x)) \cup \Phi^{-1}(R(y))$ dissects K_a into two nontrivial pieces. Otherwise almost all rays converging to one of the two arcs in S^1 between x and y would have Φ^{-1}-images converging to one point a. This contradicts the Fatou theorem. Thus take $x \neq y$ in X. $\Phi^{-1}(R(x)) \cup \Phi^{-1}(R(y))$ dissects K_a, so does the $f^{m(a)}$-image, hence $g(x) \neq g(y)$.

We have proved that the set X is finite, that is, the set $\Lambda_\tau(a)$ is a finite union of the closed sets $\lambda(x)$, $x \in X$, and, hence, also closed.

PROOF OF A PART OF 2^o: Consider an arbitrary $t \in \Lambda_\tau(a)$ for which $\omega(t)$ is infinite. Consider the restriction of the (expanding) external map $h_f \colon S^1 \to S^1$ to the compact $\omega(t)$. The map $h_f \colon \omega(t) \to \omega(t)$ is well defined, continuous and "onto". As we have assumed that $\omega(t)$ is infinite, it follows by Lemma 3.2 that the map $h_f \colon \omega(t) \to \omega(t)$ is not a homeomorphism, that is, there are two points t and t' in $\omega(t)$ such that $h_f(t) = h_f(t')$. As $f^{m(a)}$ is a local homeomorphism around a, it implies that the τ-rays with the arguments t and t' have a common line, which is possible if and only if an iterate of t and t' are arguments of a critical point of f in $W \setminus K(f)$. By $h_f(t) = h_f(t')$ this is in fact a critical point of $f^{m(a)}$. A part of the assertion 2^o of Theorem 1 is proved, we postpone the rest to the final part of Proof of Theorem 1.

Remark 3.2: We could not refer to Douady's Lemma 3.2 to prove that the set X in 1^o is finite because it was not known a priori that X was compact.

To proceed further and to be able to refer to Lemma 3.1 we need to know that (ii) holds. But this is the case, namely we have following

LEMMA 3.3: *If a is an f-periodic point in $J(f)$, then each iterate h_f^n, $n > 0$ of the external mapping h_f restricted to $\Lambda_\tau(a)$ is at most 2-to-1. In particular it cannot happen that $t \neq t'$ are in $\Lambda_\tau(a)$, $h_f(t) = h_f(t')$ and t is periodic under h_f.*

Proof: If t, t', t'' are different elements of $\Lambda_\tau(a)$ with the same image under h_f^n, then by the fact that f is a homeomorphism around a, the τ-rays corresponding to

t, t', t'' landing at a have a common line l going up to a. This is impossible because the line l has only two "sides". The map $\hat{B}$ from Section 1 is a homeomorphism from A^* cut along needles (including the banks of the needles) to $W \setminus K(f)$ cut along τ-cuts.

If $h_f^n(t) = h_f^n(t') = t \neq t'$, then for an arbitrary $t_* \in h_f^{-n}(t') \cap \Lambda_\tau(a)$ we have t_* different from t and t' because the h_f^n-images are different. Meanwhile $h_f^{2n}(t) = h_f^{2n}(t') = h_f^{n+1}(t_*)$. This contradicts the first assertion in Lemma 3.3.

Proof of 3^o: As $K_a \neq \{a\}$ there exists $b \in K_a \setminus \{a\}$ accessible by an external ray R. To see this consider, for example, an arbitrary point $z \in S^1$, not in X from the proof of the assertion 1^o, such that a K_a-related ray $\tilde{R}$ landing at z has Φ^{-1}-image R converging to b. (In fact we do not use later the convergence of $\Phi^{-1}(\tilde{R})$.) Write $x_0 = \arg_{B_f^\tau} R$. Then for $A := \Lambda_\tau(a)$, $h := h_f^{m(a)}$ and m the period of the points in X under g, the condition (i) of Lemma 3.1 holds. This is easily visible in Φ coordinates, with the use of the Jordan theorem. Condition (ii) holds by Lemma 3.3. Thus Lemma 3.1 yields the assertion 3^o.

Proof of 4^o: 4^o follows immediately from Lemma 3.1 because finitness of $\Lambda_\tau(a)$ implies the existence of an h_f periodic point in $\Lambda_\tau(a)$ which plays the role of x_0.

Continuation of the proof of 2^o: Let Λ be an invariant minimal set in $\Lambda_\tau(a)$ in the dynamics sense, i.e. every forward orbit in it is dense. Λ is infinite because $\Lambda_\tau(a)$ is infinite, hence it has no periodic orbits by 4^o. So Λ is perfect, i.e. a Cantor set. We shall end the proof if we show that actually $\Lambda = \Lambda_\tau(a)$. By the minimality h_f maps Λ onto Λ. By the monotonicity of h_f on Λ the set $S^1 \setminus \Lambda$ can be decomposed into the union of sequences of open arcs $I_{j,n}, n = 0, 1, \dots$ such that h_f maps the ends of $I_{j,n}$ to the ends of $I_{j,n-1}$ for $n > 0$ and the pair of the ends of each $I_{j,0}$ is a critical pair. We do not have any arc in $S^1 \setminus \Lambda$ wandering forward, i.e. with ends never collapsed to one point. Otherwise we would have a contradiction with the expanding property. We remark that no arc can be periodic (in the sense of having periodic ends); we have excluded it already.

If it happened that $t_0 \in I_{j,n} \cap \Lambda_\tau(a)$, then for the ends s, s' of $I_{j,n}$ we have $h_f^{n+1}(s) = h_f^{n+1}(t_0) = h_f^{n+1}(s')$. This is not possible by Lemma 3.3.

(Note that if u, u' are ends of an $I_{j,0}$ and $v \in I_{j,0}$, then $h_f(v) \neq h_f(u) = h_f(u')$ does not contradict the monotonicity if we take into account only u, v, u'. To obtain a contradiction take into account a fourth point outside $\overline{I_{j,0}}$. Another idea is to again involve the map f. Just observe that $I_{j,0} \cap \Lambda_\tau(a) = \emptyset$. Every τ-ray with the argument in $I_{j,0}$ goes to a component of $K(f)$ different from K_a.)

The proof of Theorem 1 (1^τ) is complete.

Remark 3.3: We could lift the map $h_f^{m(a)}$ on $\Lambda_\tau(a)$ to a monotone map on its preimage in $\mathbb{R}$. This allows one to consider a rotation number, rational or irrational depending on whether $\Lambda_\tau(a)$ is finite or infinite. It seems interesting to understand such sets and to study fixed point portraits depending on the combinatorics of these sets, cf. [GM].

Appendix

1. From a polynomial-like to an external map via cutting and glueing.

Let $f\colon W_1 \to W$ be the polynomial-like mapping from Section 1. Starting with the foliation τ we construct the external map h_f in a different way, via the hedgehog.

Recall that A_τ is defined as a subset of $W \setminus K(f)$ formed by the union of all the τ-radii, i.e. A_τ is $W \setminus K(f)$ with all the τ-cuts deleted. Let

$$K_\tau = W \setminus A_\tau \ , \ J_\tau = \partial K_\tau$$

denote the filled-in Julia set and, respectively, the Julia set, completed by the τ-cuts. These sets are closed.

Recall also (Section 1) that A_τ is homeomorphic to an annulus.

Thus there exists a conformal isomorphism $\varphi\colon A_\tau \to \tilde{A} = \{z\colon 1 < |z| < e^{w_\tau}\}$, where w_τ is the modulus of A_τ. The map $f\colon A_{1,\tau} = A_\tau \cap W_1 \to A_\tau$ induces a conjugated map $\tilde{f} = \varphi \circ f \circ \varphi^{-1}$ of an open subannulus $\tilde{A}_1 \subset \tilde{A}$, such that S^1 is the common inner boundary of $\tilde{A}_1$ and $\tilde{A}$. The inner boundary J_τ of A_τ is the union of Σ, which is the union of all the τ-cuts, and the Julia set $J(f)$. Since every point of Σ belongs to finitely many smooth curves and is not in $J(f)$, the map φ extends to a homeomorphism of the space $\hat{\Sigma} = \{(z,l)\colon z \in \Sigma,\ l$ is a class of equivalent (homotopic) curves in A_τ with the common end at $z\}$ onto an open subset $\tilde{\Sigma}$ of S^1, [G]. The conjugated map $\tilde{f}$ extends continuously to S^1. We will denote the extended maps by the same symbols φ and $\tilde{f}$. These definitions are illustrated in Figure A1.

LEMMA A.1: *Let $\tilde{\alpha}$ be a component of $\tilde{\Sigma}$. There exists a unique point $\tilde{q} = \tilde{q}(\tilde{\alpha}) \in \tilde{\alpha}$ which splits the arc $\tilde{\alpha}$ into two arcs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, such that:*

(a) *For $i = 1, 2$, if $x, y \in \tilde{\alpha}_i$, $x \neq y$, then $\tilde{f}^n(x) \neq \tilde{f}^n(y)$ for all positive n (whenever $\tilde{f}^n(x), \tilde{f}^n(y)$ are in $\tilde{A} \bigcup S^1$).*

(b) *There exists a homeomorphism $\pi_{\tilde{\alpha}}\colon \tilde{\alpha}_1 \to \tilde{\alpha}_2$ such that for every $x \in \tilde{\alpha}_1$ and for some positive integer m depending on x,*

$$\tilde{f}^m(x) = \tilde{f}^m(\pi_{\tilde{\alpha}}(x)) \in \tilde{A}.$$

This equality holds precisely for such m that $\tilde{f}^m(x) \in \tilde{A}$, i.e. $\notin S^1$.

(c) *For $x \in \tilde{\alpha}_1$, if $\tilde{f}^n(x) \in S^1$ then $\tilde{f}^n(x) \in \tilde{\beta}$ a component of $\tilde{\Sigma}$, $\tilde{f}^n(\pi_{\tilde{\alpha}}(x)) \in \tilde{\beta}$ and $\tilde{f}^n \pi_{\tilde{\alpha}}(x) = \pi_{\tilde{\beta}} \tilde{f}^n(x)$.*

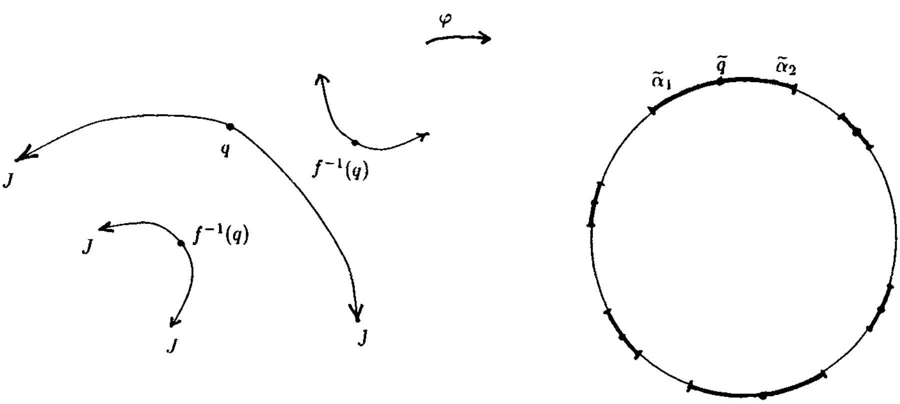

Figure A1. Cutting along τ-cuts.

Proof: Consider the continuous arc $\alpha = \varphi^{-1}(\tilde{\alpha}) \subset \hat{\Sigma}$. Denote the projection of $\hat{\Sigma}$ to Σ by P. A positive direction on α induced by the positive direction given on $P(\alpha)$ (given by the orientation of τ towards $J(f)$) is defined at every point of α, except for the points of $P^{-1}(C_f(\infty))$, at which the direction can change. Let us go along a part of α in the negative direction, starting from a point $(z, l) \in \alpha \setminus P^{-1}(C_f(\infty))$. Then we must arrive along $P(\alpha)$ at a point $q \in C_f(\infty)$ at which the direction changes. Let us prove that q is a unique point of $P(\alpha)$ with this property. Otherwise there exists a point $q_1 \in P(\alpha) \cap C_f(\infty)$ which is a terminal point for two consecutive τ-cuts $P(\alpha')$ and $P(\alpha'')$ in $P(\alpha)$. Then we have two cases:

If $P(\alpha') \neq P(\alpha'')$ (see Figure A2), there are other τ-cuts β_1, β_2 with the initial point q_1, such that β_1 and β_2 lie on the different sides of the curve $P(\alpha)$ (remark that at every point c of $C_f(\infty)$ the directions of the τ-lines with the end point c alternate to each other). It contradicts the continuity of $\alpha \in \hat{\Sigma}$.

If $P(\alpha') = P(\alpha'')$ (we go along them in opposite directions) already the existence of one τ-cut starting at q_1 contradicts the continuity of α.

We have proved that the point $P^{-1}(q)$ splits the curve α into two curves α_1 and α_2, such that the direction on each of them does not change. In particular P is one-to-one on each of them.

For each $i = 1, 2$, if $x, y \in P(\alpha_i)$ and $x \neq y$, then $f^n(x) \neq f^n(y)$, for all positive n for which the images are defined. This is so, for example, because M (Definition 1.2), hence $M \circ f^n$, is strictly monotone on each α_i. On the other hand f maps the τ-lines to the τ-lines, and we have $q \in C_f(\infty)$. Hence, for every $z \in \alpha_1$ there exist a unique point $Z \in \alpha_2$ and $m \in \mathbb{N}$, such that $f^m(P(z)) = f^m(P(Z)) \in A_\tau$, and the correspondence $z \to Z$ is a homeomorphism $\alpha_1 \to \alpha_2$. This induces the homeomorphism $\pi_{\tilde{\alpha}}$: $\tilde{\alpha}_1 \to \tilde{\alpha}_2$, where $\tilde{\alpha}_i = \varphi(\alpha_i)$. The point $\tilde{q} = \varphi(P^{-1}(q))$ is the common beginning of both $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

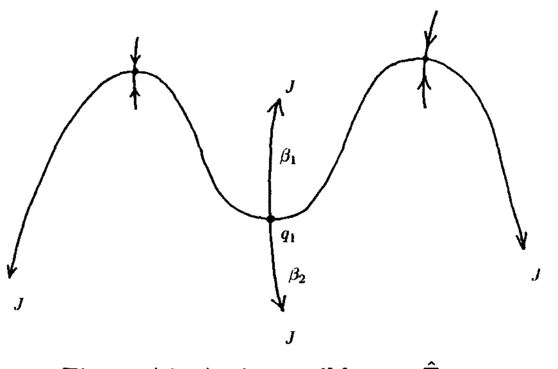

Figure A2. An impossible $\alpha \subset \hat{\Sigma}$.

Observe finally that $\tilde{f}^m(x) \in \tilde{A}$ if and only if $P\varphi\tilde{f}^m(x)$ belongs to a τ-radius (i.e. it is already not in a τ-cut). If we go backward and then forward along a τ-radius to the same level of the function M, the points at which we start and end coincide. We remark that the map $\pi_{\tilde{\alpha}}$ defined here is just a mapping preserving the M-values (see Remark 1.1). This concludes the proof of (b).

To prove (c) observe that $\varphi^{-1}\tilde{f}^n(\tilde{\alpha}_1)$ is the union of a continuous curve b_1 in $\hat{\Sigma}$ (the concatenation of a sequence of P-preimages of τ-cuts) and a part R of a τ-radius. So $\varphi^{-1}\tilde{f}^n(\tilde{\alpha}_2)$ is also a union of b_2 in $\hat{\Sigma}$ and the same R. We set $\tilde{\beta} = \varphi(b_1 \cup b_2)$. Lemma A.1 is proved.

We are ready to give the following definitions. Recall that $\tilde{A}_1$ is a subannulus of $\tilde{A}$ with the same inner boundary S^1, and $\tilde{\Sigma} \subset S^1$. Let A^* and $A_1^* \subset A^*$ be Riemann surfaces formed from the open annuli $\tilde{A}$ and $\tilde{A}_1$ as follows: for every component $\tilde{\alpha}$ of the set $\tilde{\Sigma} \subset \partial\tilde{A}$ let us glue every point $x \in \tilde{\alpha}$ with the point $\pi_{\tilde{\alpha}}(x)$. Let

$$\Pi: \tilde{A} \to A^*$$

be the projection. Consider

$$U_\tau = \Pi(\tilde{A}),$$
$$U_{1,\tau} = \Pi(\tilde{A}_1)$$

the surfaces with the complex structure σ_0 induced by the standard one from $\tilde{A}$, and let

$$h_f: U_{1,\tau} \to U_\tau$$

be the holomorphic map induced by $\tilde{f}: \tilde{A}_1 \to \tilde{A}$. The projection Π is defined on $\tilde{\Sigma}$ as well.

The Riemann surface U_τ is called the **hedgehog-like annulus.** The inner boundary S_τ of U_τ is called the **hedgehog.** The set $\Pi(\tilde{\Sigma})$ is the set of the hedgehog's **needles.**

THEOREM A.1: *σ_0 extends to a unique complex structure on A^*. The map h_f extends to an analytic unbranched covering map of the A_1^* onto the A^* of degree d. The Riemann surface A^* is conformally isomorphic to an annulus*

$$\{z: 1 < |z| < e^r\}, \quad 0 < r < \infty.$$

Denote h_f transported to this annulus also by h_f. Then the restriction to S^1 of the extension of h_f beyond S^1 is the external map of the polynomial-like mapping f.

Proof: The map $h_f: U_{1,\tau} \to U_\tau$ is holomorphic and has no critical points. By the construction, h_f extends to a continuous map of A_1^* into A^* because, if points $x, y \in \tilde{\Sigma}$ are glued together, the points $\tilde{f}(x)$ and $\tilde{f}(y)$ are either also glued or coincide and lie in $\tilde{A}$ (Lemma A.1 (b), (c)). Moreover, the extended map h_f is a local homeomorphism, because the map $\tilde{f}: \tilde{A}_1 \to \tilde{A}$ is a local homeomorphism.

To see that h_f maps points $x, y \in A_1$, $x \neq y$, close to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ respectively, to different points, observe that $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$ are in τ-radii on different sides of the τ-radius ending at $P\varphi^{-1}(\tilde{q}(\tilde{\alpha}))$. The f^m-images (m from Lemma A.1

(b)) of these radii cannot hit the same τ-radius if they are close enough to each other. The same consideration concerns $x \in \tilde{\alpha}$, $y \in \tilde{A}$ close to $\tilde{\alpha}$ or x, y close to $\tilde{q} \in \tilde{\alpha}$.

Let N be a small neighborhood of a point in a needle so that, for some positive n, $H = h_f^n\colon N \to M$ is a homeomorphism and the image M is a subset of a neighbourhood of $A^* \setminus A_1^* \subset U_\tau$. Let F be an arbitrary complex-valued function continuous in N and holomorphic in $N \bigcap U_{1,\tau}$. The function $F \circ H^{-1}$ is continuous in M and holomorphic there except for a smooth arc. This is a removable singularity for such a function. Hence, $F \circ H^{-1}$ extends to a holomorphic function in M. It means that A^* admits precisely one complex structure which is an extension of that from U_τ, and such that this structure coincides with the preimage of the complex structure from a neighbourhood of $A^* \setminus A_1^*$ under the local homeomorphisms h_f^{-n}. It is obvious now that the map h_f is analytic in A_1^*. Moreover, it is an unbranched map of A_1^* onto A^*. It is easy to see now that A_1^* and A^* are annuli and $h_f\colon A_1^* \to A^*$ is biholomorphically conjugate to the identically denoted map from Section 1 giving an external map.

Remark A.1: We have proved additionally that every needle of the hedgehog S_τ is a smooth curve in A^*.

Remark A.2: As mentioned above we have constructed again an external map. So, up to an analytic conjugacy, it is independent of the foliation τ and the angle τ, though the construction of $h_f\colon A_1^* \to A^*$ depended on τ. Let us explain it in more detail.

We identify A^* with a standard annulus $\{z\colon 1 < |z| < e^r\}$. A_1^* is a subannulus with the common inner boundary S^1, $h_f\colon A_1^* \to A^*$ (hence Γ_f^* consists of smooth curves). The conformal map

$$B_f^\tau = \Pi \circ \varphi\colon A_\tau \to U_{1,\tau}$$

conjugates $f|_{A_{1,\tau}}$ and $h_f|_{U_{1,\tau}}$.

Denote the objects introduced to define an external map in Section 1 (to make a distinction with the objects here) by $\hat{h}_f\colon \hat{A}_1^* \to \hat{A}^*$, $\hat{B}_f^\tau$. The maps h_f and $\hat{h}_f$ are conjugate on neighbourhoods of $A^* \setminus A_1^*$ and $\hat{A}^* \setminus \hat{A}_1^*$ by the map $\alpha = \hat{B}_f^\tau \circ (B_f^\tau)^{-1}$. We extend this conjugacy by $\hat{h}_f^{-n} \circ \alpha \circ h_f^n$ (an analytic continuation) towards S^1 and then beyond it. Thus indeed h_f is real-analytically conjugate to the external map $\hat{h}_f$ from Section 1.

Observe that the same proof shows that the external map is indeed independent (again up to real-analytic conjugacy) of the choice of $W_1 \subset W$ in the definition of the polynomial-like map.

Observe finally that if h_f is fixed, the map B_f^τ is defined up to an analytic homeomorphism of S^1 commuting with

$$h_f\colon S^1 \to S^1.$$

2. From an external to a polynomial-like map.

We end this Appendix and the paper with a construction, which is inverse with respect to the above one in degree two (cf. [Go, Proposition 3.8]).

Fix a real analytic mapping h of degree two of the unit circle S^1, with a neighborhood E of analyticity. We assume $h^{-1}(E)$ is a proper subset of E, and E is symmetric with respect to S^1. Denote by A^* the part of E outside the unit disc, and fix a point w_* in A^*. Given h and w_*, we construct a corresponding quadratic-like mapping (up to an analytic conjugacy in a neighborhood of its Julia set) with the **disconnected Julia set** as follows:

Let l be a continuous rectifible curve in A^* joining w_* to a point w_0 on S^1. We assume that the point w_0 is not periodic for h. We call the curve l **admissible** if all the preimages of l under all the iterates of h are pairwise disjoint. Let $\Sigma_h(l) := \bigcup_{n>0} h^{-n}(l)$. Then there is a conformal isomorphism $\pi = \pi_l$ of $A^* \setminus \Sigma_h(l)$ onto a round annulus $\widetilde{A} = \{w\colon 1 < |w| < r\}$, and h induces a map $\widetilde{h}$ in $\pi(h^{-1}(A))$, which extends to its inner boundary S^1. The set $\pi(\Sigma_h(l))$ is also well defined. It is a collection of arcs on S^1. There exists a unique equivalence relation $\sim$ between the points of these arcs, which obey the following two properties: (1) if $x \sim y$, then if $x \neq y$ they belong to two different arcs and, for an adequate $n > 0$, $\widetilde{h}^n(x) = \widetilde{h}^n(y) \in \pi(l)$; (2) the relation $\sim$ can be extended to the unit disc (by geodesics in the Poincaré metric) without intersections, i.e. $\sim$ is a **lamination**, see Figure A3. The existence and uniqueness follow from [Th].

Now we contract each leaf of the lamination to a point, in particular we glue the pairs of points of the arcs $\pi(\Sigma_h(l))$ according to the relation $\sim$, and obtain a Riemann surface S, which inherits a complex structure from $\widetilde{A}$. The surface S is a **planar Riemann surface** since every closed loop in it separates it. By the Uniformization Theorem, we can consider S as a domain in $\mathbb{C}$ with the standard complex structure. The projection $\psi\colon \widetilde{A} \to S$ is a conformal isomorphism onto the image $S' \subset S$. The complement $\Sigma = S \setminus S'$ consists of open arcs corresponding to the arcs from $\pi(\Sigma_h(l))$. Note also that $\widetilde{h}$ induces a holomorphic map f on the part S' of S.

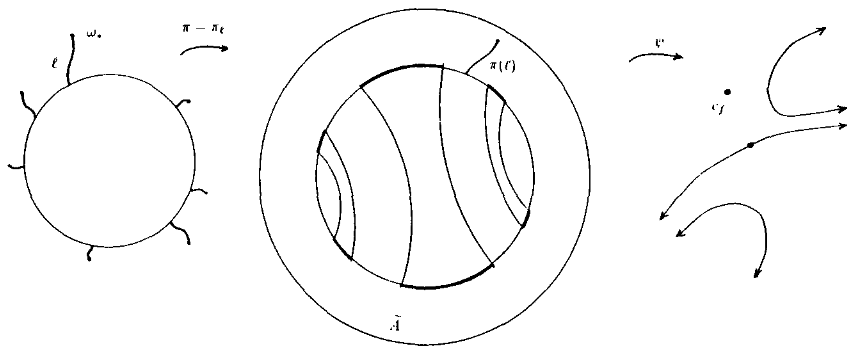

Figure A3. Thurston's lamination.

Denote by J the union of the bounded components of the complement $\mathbb{C} \setminus S$. A **priori**, it is a union of points and discs. Nevertheless discs are not possible because we obtain a standard picture: each component of J is the intersection of a centered sequence of holomorphic discs D_n so that all $D_n \setminus \mathrm{cl}\, D_{n+1}$ have positive moduli bounded away from 0.

We even obtain that J has *absolute measure zero* (repeating an argument from [McM]), that is, $\widetilde{S}$ is removable for the holomorphic maps outside this compact set J (see [AB]).

This removability of J allows us to extend the map f to a holomorphic one in a simply connected domain $W_1 := S \cup J$. Thus f is a quadratic-like mapping on W_1 with the *disconnected Julia set* J.

The map f (more exactly, a class of conformally conjugated maps) depends on the point w_*, but does not depend on the initial arc l (whenever it is admissible). Indeed, let l_1 be another arc in A joining w_* and a point $w_1 \in S^1$, and f_1 be a quadratic-like map obtained in the above construction with the curve l_1 instead of l. Using the fact that f and f_1 are holomorphically conjugate by $H := \psi_{l_1} \circ \pi_{l_1} \circ \pi_l^{-1} \circ \psi_l^{-1}$ in annuli containing the critical values $c_f = \psi_l \pi_l(w_*)$ and $c_{f_1} = \psi_{l_1} \pi_{l_1}(w_*)$ respectively, so that $H(c_f) = c_{f_1}$, it is easy to see (extending the conjugacy by $(f^n)^{-1} \circ H \circ f^n$ and again using the removability of J) that then the map f_1 is holomorphically conjugate to f in their domains of definition.

Thus, we have constructed a space Q_h of quadratic-like mappings with the prescribed external map h. This operation is inverse to the previous construction of an external map. So it allows us to introduce a complex structure of A^* in the space Q_h. If $h(w) = w^2$, then Q_h is the space of quadratic polynomials $f_c(z) = z^2 + c$ with disconnected Julia sets, the annulus A^* is a punctured disc, and we obtain the Douady–Hubbard theorem: the Mandelbrot set is connected. More precisely, there is a one-to-one correspondence Ψ between the points of the punctured disc and the parameters c outside the Mandelbrot set. Furthermore, this correspondence Ψ is a holomorphic map since, by the construction, its inverse coincides with the Douady–Hubbard map $B_c(c)$, where each $B_c(z)$ conjugates f_c with $z \mapsto z^2$ in a neighborhood of infinity containing c [DH1].

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